Exact solution for a one-dimensional model for reptation

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We discuss the exact solution for the properties of the recently introduced "necklace" model for reptation. The solution gives the drift velocity, diffusion constant, and renewal time for asymptotically long chains. Its properties are also related to a special case of the Rubinstein-Duke model in one dimension.

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I. INTRODUCTION

The most common lattice model for reptating polymers is the Rubinstein-Duke model [1,2]. The major drawback of this model is that it has thus far defied an exact solution. Therefore, it is natural to look for models that are in the same spirit, but that are more open to analytic rather than numerical methods. Recently Guidoni *et al.* [3] introduced and analyzed a one-dimensional model for reptation, to which we shall refer as the "necklace model." It is a chain that moves through the exchange of beads and vacancies along a line. The purpose of this note is to show that the model permits an exact solution by means of the matrix product expansion, due to Derrida *et al.* [4]. Also the spectrum of the master operator can be derived, leading to an exact expression for the renewal time. In addition we relate the properties to those of a special one-dimensional variant of the Rubinstein-Duke model with hernia creation and annihilation.

II. THE NECKLACE MODEL

The necklace model is a string of $N+1$ beads located on a line of points. The beads are either neighbors or nearest neighbors. In the latter case, there is an unoccupied lattice site (a vacancy) between the beads. The beads are not allowed to occupy the same lattice point, and two consecutive vacancies are forbidden in order to ensure the integrity of the chain. The possibility of vacancies is an expression of the elasticity of the chain. The internal beads hop by exchanging with a vacancy. The two end beads can also exchange with a vacancy from outside [see Fig. 1(a)]. Clearly the number of beads is conserved, but not the number of vacancies, which enter and leave at the ends and migrate through the chain. The stationary state of the system follows from the master equation governing this stochastic process.

The key to the exact solution is to focus on the motion of the vacancies rather than on that of the beads. There are N positions for the vacancies available, since each vacancy must be surrounded by beads. Each of the N positions can be occupied by a vacancy or not. The state of the chain of beads is fully determined by the occupation distribution of the vacancies. Thus, we consider the chain as an open system for vacancies, which hop with the rates derived from the motion of the original beads. Their hopping is constrained by the rule that two vacancies cannot occupy the same position. It would mean that the two beads surrounding such a double vacancy would be separated by two vacancies.

Transfering the motion for the beads to the vacancies we arrive at the following rules: (i) An internal vacancy can hop to a neighboring empty position with rate p_c to the right and p_c' to the left; (ii) at the left hand end of the chain, a vacancy enters with rate p_a and leaves with rate p_a'; and (iii) on the right-hand end, the rates are p_b for leaving and p_b' for entering.

Note that we have primed rates for the hops to the left and unprimed for the motion to the right. The asymmetry between left and right can be attributed to a driving field on the beads, inducing a bias B for a hop to the left and a bias B^{-1} for a rightward hop. These biases may derive from a charge on the beads, which is influenced by an electric field. Then we have for the ratios the relations

$$\frac{p_a'}{p_a} = \frac{p_b'}{p_b} = \frac{p_c'}{p_c} = B^2. \tag{1}$$

For the solubility of the model, it is not neccessary to assume these ratios and only when we discuss the properties in more detail will we use this physical restriction. The above given rates are more general than those used by Guidoni *et al.*; in particular, we allow for a finite driving field. For our exact solution, it is however important that the hopping rules for the internal vacancies are uniform along the chain.

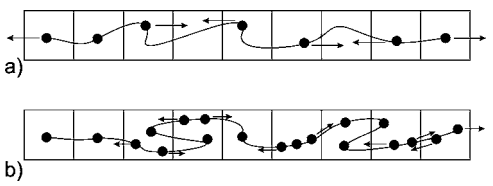

FIG. 1. Comparison of the moves in (a) the necklace model and (b) the Rubinstein-Duke model. The arrows indicate the beads (reptons) that can move.

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III. THE MASTER EQUATION

The state of the chain can be represented by N variables τ_i, which assume the values 1 and 0, where 1 viz. 0 corresponds to the presence viz. absense of a vacancy. The probability distribution $P(\tau_1, \ldots, \tau_N)$ of the stationary state follows from the master equation

$$\mathcal{M}P(\tau_1, \ldots, \tau_N) = 0, \tag{2}$$

with $\mathcal{M}$ the master operator. It contains the usual gain term consisting of all transitions, which increase the probability $P(\tau_1, \ldots, \tau_N)$, and a loss term, which contains all transitions out of the configurations $(\tau_1, \ldots, \tau_N)$. The master operator is the sum of $N+1$ operators, representing the action of the beads

$$\mathcal{M} = \sum_{i=0}^{N} \mathcal{M}_i. \tag{3}$$

As a first example, consider the bead at the left hand of the chain, leading to $\mathcal{M}_0$. It acts on the τ_1 dependence of $P(\tau_1, \ldots, \tau_N)$, as it only influences the existence of a vacancy on the first position. It is given by the expression

$$\mathcal{M}_0 P(\tau_1 \cdots) = (1 - \tau_1)[p_a' P(1 \cdots) - p_a P(0 \cdots)] + \tau_1[p_a P(0 \cdots) - p_a' P(1 \cdots)]. \tag{4}$$

The first two terms refer to the case where a vacancy is absent from the first possible position. It has a gain and a loss term. Similarly, the last two terms refer to the case where a vacancy is present. The expression can be shortened to

$$\mathcal{M}_0 P(\tau_1 \cdots) = (1 - 2\tau_1)[p_a' P(1 \cdots) - p_a P(0 \cdots)]. \tag{5}$$

In the same way, the other parts of the master operator are expressed as

$$\begin{cases} \mathcal{M}_j P(\cdots \tau_j, \tau_{j+1} \cdots) = (\tau_j - \tau_{j+1})[p_c' P(\cdots 0,1 \cdots) - p_c P(\cdots 1,0 \cdots)] \\ \mathcal{M}_N P(\cdots \tau_N) = (1 - 2\tau_N)[p_b P(\cdots 1) - p_b' P(\cdots 0)]. \end{cases} \tag{6}$$

This form of the master operator shows that it has an eigenvalue 0, as summation over all τ_i yields zero for all individual terms, implying conservation of probability. The left eigenvector is constant for all configurations, but the right eigenvector is the nontrivial probability distribution of the stationary state.

IV. MATRIX PRODUCT EXPANSION FOR THE STATIONARY STATE

The necklace model is a so-called asymmetric simple exclusion process (ASEP). The literature on the ASEP's is extensive and their connection with reptation has been noted earlier [5]. The stationary state can be found by a matrix product expansion, in which the probability distribution $P(\tau_1, \ldots, \tau_N)$ is represented as a product of matrices

$$P(\tau_1, \ldots, \tau_N) = Z_N^{-1} \langle W | \prod_{j=1}^{N} [\tau_j \mathcal{D} + (1 - \tau_j)\mathcal{E}] | V \rangle. \tag{7}$$

As we shall see, $\mathcal{D}$ and $\mathcal{E}$ are sort of creation and annihilation operators, represented by infinite dimensional matrices. $|V\rangle$ and $\langle W|$ are states in this space. The normalization is given by

$$Z_N = \langle W | (\mathcal{D} + \mathcal{E})^N | V \rangle. \tag{8}$$

The matrices $\mathcal{D}$ and $\mathcal{E}$, as well as the states $\langle W|$ and $|V\rangle$ are to be determined such that (7) is the stationary state of the master equation. By insertion of (7), we find the action of the $\mathcal{M}_j$ on the factors

$$\begin{cases} \mathcal{M}_0[\tau_1 \mathcal{D} + (1 - \tau_1)\mathcal{E}] = (1 - 2\tau_1)(p_a' \mathcal{D} - p_a \mathcal{E}), \\ \mathcal{M}_j[\tau_j \mathcal{D} + (1 - \tau_j)\mathcal{E}][\tau_{j+1} \mathcal{D} + (1 - \tau_{j+1})\mathcal{E}] = (\tau_j - \tau_{j+1})(p_c' \mathcal{E}\mathcal{D} - p_c \mathcal{D}\mathcal{E}), \\ \mathcal{M}_N[\tau_N \mathcal{D} + (1 - \tau_N)\mathcal{E}] = (1 - 2\tau_N)(p_b \mathcal{D} - p_b' \mathcal{E}). \end{cases} \tag{9}$$

Then we use the identity

$$(\tau_j - \tau_{j+1})(\mathcal{D} + \mathcal{E}) = (1 - 2\tau_{j+1})[\tau_j \mathcal{D} + (1 - \tau_j)\mathcal{E}] - (1 - 2\tau_j)[\tau_{j+1} \mathcal{D} + (1 - \tau_{j+1})\mathcal{E}] \tag{10}$$

and impose the condition

$$p_c'\mathcal{E}\mathcal{D} - p_c\mathcal{D}\mathcal{E} = \zeta[\mathcal{D} + \mathcal{E}], \tag{11}$$

in order to rewrite the expression for the internal reptons as

$$\begin{aligned}\mathcal{M}_j[\tau_j\mathcal{D} + (1-\tau_j)\mathcal{E}][\tau_{j+1}\mathcal{D} + (1-\tau_{j+1})\mathcal{E}] \\ = \zeta(1-2\tau_{j+1})[\tau_j\mathcal{D} + (1-\tau_j)\mathcal{E}] - \zeta(1-2\tau_j) \\ \times[\tau_{j+1}\mathcal{D} + (1-\tau_{j+1})\mathcal{E}].\end{aligned} \tag{12}$$

The "commutation" relation (11) defines the properties of the $\mathcal{D}$ and $\mathcal{E}$ matrices. The states $|V\rangle$ and $\langle W|$ are fixed by the relations

$$\langle W|(p_a'\mathcal{D} - p_a\mathcal{E}) = \langle W|\zeta, \quad (p_b'\mathcal{E} - p_b\mathcal{D})|V\rangle = \zeta|V\rangle. \tag{13}$$

Then it is possible to recombine in (9) the factors again to probabilities

$$\begin{cases} \mathcal{M}_0 P(\tau_1, \ldots, \tau_N) = \zeta(1-2\tau_1)P(\tau_2, \ldots, \tau_N) \\ \mathcal{M}_j P(\tau_1, \ldots, \tau_N) = -\zeta(1-2\tau_j)P(\tau_1, \ldots, \tau_{j-1}, \tau_{j+1}, \ldots, \tau_N) + \zeta(1-2\tau_{j+1})P(\tau_1 \ldots, \tau_j, \tau_{j+2}, \ldots, \tau_N) \\ \mathcal{M}_N P(\tau_1, \ldots, \tau_N) = -\zeta(1-2\tau_N)P(\tau_1, \ldots, \tau_{N-1}). \end{cases} \tag{14}$$

The parameter ζ is arbitrary; it has been introduced to facilitate the normalization. The sum over all these relations vanishes, which shows that (7) is the stationary state of the master equation.

Relation (11) can be clarified by the substitution

$$\mathcal{D} = 1 + b^\dagger, \quad \mathcal{E} = 1 + b. \tag{15}$$

This yields for the creation and annihilation operators $b^\dagger$ and b the q-deformed commutation relation

$$bb^\dagger - qb^\dagger b = 1 - q, \tag{16}$$

provided that we fix ζ and q as

$$\zeta = p_c' - p_c, \quad q = p_c/p_c'. \tag{17}$$

The advantage of the new operators is that their spectrum has been thoroughly investigated. They have similar properties as the usual creation and annihilation operators for the harmonic oscillator. Note that for $q=1$ the operators seem to commute, but it is easy to rescale $b^\dagger$ and b such that the right-hand side of (16) becomes 1, as often is done [7].

The power of the representation is that it gives the probability distribution for all chain lengths N. The properties of the matrices $\mathcal{D}$ and $\mathcal{E}$ and the states $\langle W|$ and $|V\rangle$ are independent of N. It requires, however, quite a bit of formal manipulation to retrieve the chain properties from the general expression (7).

The relations (11) and (13) were mentioned by Derrida *et al.* [4]. They were related to the q-deformed algebra by Sasamoto [6] and Blythe *et al.* [7], leading to explicit formulas. Unfortunately, the case that we see as most physical [relations (1) and all intrinsic mobilites the same], does not fall in the wide range of parameters already treated. However, the technique that they employ can be used to deal with our case. As it is not our aim to marginally extend the exact solutions to an even wider regime, we refer to [6,7] for details.

V. THE DRIFT VELOCITY

The phase diagram as a function of the parameters is governed by the magnitudes of the net hopping rates

$$\Delta p_a = p_a' - p_a, \quad \Delta p_b = p_b' - p_b, \quad \Delta p_c = \tfrac{1}{2}(p_c' - p_c). \tag{18}$$

The smallest is the limiting factor. If the input on the left-hand side is small, a dilute phase results. If the output at the right-hand side is small, a dense phase will form. If the throughput in the bulk of the chain is the limiting factor, a maximum current phase appears, with a constant density of vacancies in the bulk.

For illustration, we discuss the case (1), with hopping rates only influenced by the driving field with bias B. As an overall rate sets the time scale, we may write

$$p_a' = p_b' = p_c' = B, \quad p_a = p_b = p_c = B^{-1}. \tag{19}$$

Clearly, we are then in the maximum current phase. We restrict ourselves to the most interesting property: the drift velocity. The drift of bead j, in a specific configuration, is given by

$$v_j = p_c' P(\cdots 0,1 \cdots) - p_c P(\cdots 1,0 \ldots). \tag{20}$$

The first term is the probability to jump to the right, and the second for that to the left. Remember that the beads jump in the opposite direction of the vacancies. Comparing this to relation (6), we have the identity

$$\tau_j \mathcal{M}_j P(\tau_1, \ldots, \tau_N) = \tau_j(1-\tau_{j+1})v_j. \tag{21}$$

By summing over the variables τ_j and τ_{j+1}, the factor in front of v_j gives 1, and one finds for the average drift

$$\langle v_j \rangle = \langle \tau_j \mathcal{M}_j \rangle. \tag{22}$$

This average can be expressed, with the aid of (14), as

$$v = Z_N^{-1}\langle W|(\mathcal{D}+\mathcal{E})^{N-1}V\rangle = \frac{Z_{N-1}}{Z_N}\zeta. \tag{23}$$

We have dropped the index j as the result is obviously independent of j. It has to, since there is no accumulation of beads in the stationary state. Thus, (23) is the expression for the drift velocity, relating it to the normalization factors Z_N. The proportionality of the drift to $\zeta = p_c' - p_c = B - B^{-1}$ is simply a matter of time scales: when the rates go up, the drift follows proportionally.

The Z_N are expressed in terms of what is called the position operator $b^\dagger + b$ of the q-deformed harmonic oscillator

$$\mathcal{D} + \mathcal{E} = 2 + b + b^\dagger. \tag{24}$$

The spectrum of $b^\dagger + b$ has been investigated in detail [6]. It can be represented as

$$(b + b^\dagger)|\theta\rangle = 2\cos\theta|\theta\rangle. \tag{25}$$

The spectrum is continous: $0 \leqslant \theta \leqslant \pi$. Thus, we get

$$Z_N = \int_0^\pi d\theta w(\theta)[2(1+\cos\theta)]^N \langle W|\theta\rangle\langle\theta|V\rangle. \tag{26}$$

Here $w(\theta)$ is a function that enters in the closure relation for the states $|\theta\rangle$ (see [6]). For large N, the power of $1+\cos\theta$ gets strongly peaked around $\theta=0$ and one can apply the saddle point method to find the leading term. We put $\theta=0$ in all nonsingular terms yielding

$$Z_N \simeq z(q)4^N N^{-3/2}, \tag{27}$$

where the factor $z(q)$ is independent of N and contains among others the factor $\langle W|0\rangle\langle 0|V\rangle$. Fortunately, this factor drops out in the drift velocity

$$v \simeq \tfrac{1}{4}(B - B^{-1}), \quad N \to \infty. \tag{28}$$

This expression is valid for all values of $B > 1$. The (dimensionless) field ϵ enters in the bias as $B = \exp(\epsilon/2)$. Thus, the difference $B - B^{-1}$ becomes proportional to the field strength ϵ when $\epsilon \to 0$. Of course, the drift vanishes when the field vanishes.

The diffusion coefficient is obtained as the derivative of the drift velocity with respect to the force. Thus, we get

$$D \simeq \frac{(B + B^{-1})}{8N}, \quad N \to \infty. \tag{29}$$

The factor N in the denominator is due to the fact that the force acts on all the beads. Thus, we see that the diffusion coefficient decays as N^{-1} and not as N^{-2}, which is typical for reptation models. Usually one refers for the diffusion coefficient to the value in the zero field limit $B=1$.

Relations (28) and (29) show that we are in the maximum current regime, which is not limited by the input and output at the ends of the chain.

VI. SMALL CHAINS

The asymptotically leading behavior is, of course, the most interesting result, showing the strength of the matrix product expansion. However, it is also worthwhile to see that the whole behavior is determined by the commutation relation (16). To illustrate this point we confine ourselves to the physical restriction (1). Then the equations for $\langle W|$ and $|V\rangle$ reduce to

$$\langle W|b^\dagger = q\langle W|b, \quad b|V\rangle = qb^\dagger|V\rangle. \tag{30}$$

Note that $\langle W| = \langle V|$, since the first relation (30) is the conjugate of the second.

As a first example of playing with commutation relations, consider

$$\langle V|(b + b^\dagger)|V\rangle = q\langle V|(b^\dagger + b)|V\rangle, \tag{31}$$

where we used (30) to interchange b and $b^\dagger$ by applying b to $|V\rangle$ and $b^\dagger$ to $\langle V|$. As $q \neq 1$ the average has to vanish. This holds also for all odd powers of $b+b^\dagger$. The second example is the average of $b^\dagger b$, for which we first use the relations (30)

$$\langle V|bb^\dagger|V\rangle = q^{-2}\langle V|b^\dagger b|V\rangle. \tag{32}$$

But the commutation relation (16) can also be used to convert the right-hand side of (32) to the left-hand side expression. From these two relations, one deduces

$$\langle V|b^\dagger b|V\rangle = \frac{q^2(1-q)}{(1-q^3)}\langle V|V\rangle. \tag{33}$$

All other averages of two creation or annihilation operators directly follow, with (30), from this expression. Thus, the current of the two-link system equals

$$v_{N=2} = \frac{2(1+q+q^2)}{5+6q+5q^2}(B - B^{-1}), \tag{34}$$

an expression, which of course, can also be obtained by solving the probability distribution from the master equation. By this technique, it will be a long and hard road to get to the behavior at large N.

VII. THE RENEWAL TIME

Another interesting quantity is the renewal time. It is defined as the slowest time of decay toward the stationary state, and it follows from the spectrum of the master operator as the eigenvalue with the smallest negative real part. Clearly, the corresponding state decays the slowest and the renewal time is the inverse of the gap in the spectrum. The matrix product representation does not lead to the full spectrum of the master operator; it only gives the stationary state eigenfunction. On the other hand, the renewal is usually defined in the fieldless case $B=1$ for which the master operator becomes Hermitian (or symmetric in our case). The general master operator can be expressed in terms of the operators $a_j^\dagger$, creating a vacancy, and a_j annihilating a vacancy (see [9])

$$\begin{cases} \mathcal{M}_0 = B(a_1 - a_1^\dagger a_1) + B^{-1}(a_1^\dagger - a_1 a_1^\dagger), \\ \mathcal{M}_j = B(a_j^\dagger a_{j+1} - a_j a_j^\dagger a_{j+1}^\dagger a_{j+1}) + B^{-1}(a_j a_{j+1}^\dagger - a_j^\dagger a_j a_{j+1} a_{j+1}^\dagger), \\ \mathcal{M}_0 = B^{-1}(a_N - a_N^\dagger a_N) + B(a_N^\dagger - a_N a_N^\dagger). \end{cases} \tag{35}$$

The a_j are hard-core boson operators, equivalent with spin 1/2 operators, via the relations

$$a^\dagger = \frac{(\sigma^x + i\sigma^y)}{2}, \quad a = \frac{(\sigma^x - i\sigma^y)}{2}, \tag{36}$$

with the consequence

$$a^\dagger a = \frac{(1+\sigma^z)}{2}, \quad aa^\dagger = \frac{(1-\sigma^z)}{2}, \tag{37}$$

where the σ's are the Pauli matrices. The master operator, expressed in these spin operators, is an unusual Hamiltonian since it is non-Hermitian. However, for the undriven system $B=1$, it turns into the well-known ferromagnetic Heisenberg chain,

$$\mathcal{M} = \frac{1}{2}\sum_{j=1}^{N-1} [\sigma_j \sigma_{j+1} - 1] + \sigma_1^x + \sigma_N^x - 2. \tag{38}$$

Because of the magnetic field in the x direction on the boundaries of the chain, it is profitable to work in a basis of eigenstates of σ^x. These are the symmetric (spin-up) and the anti-symmetric (spin-down) combinations of a vacancy and its absence,

$$|\uparrow\rangle = 2^{-1/2}[|1\rangle + |0\rangle], \quad |\downarrow\rangle = 2^{-1/2}[|1\rangle - |0\rangle]. \tag{39}$$

The "ground state" (being the highest in the spectrum) of the chain is the state with all spins directed in the x direction, which has an eigenvalue 0. In vacancy language, this is the state in which all configurations have the same probability.

As the Hamiltonian (38) conserves the number of spins in the x direction, the spectrum breaks up into sectors with a given number of spins up. The smallest excitation from the ground state is in the sector with one spin down. Let x_n be the value of the state with the down spin at the position n. Then we have the set of equations

$$\begin{cases} -(3+\lambda)x_1 + x_2 = 0 \\ x_1 - (2+\lambda)x_2 + x_3 = 0 \\ \cdots = 0 \\ x_{N-1} - (3+\lambda)x_N = 0 \end{cases} \tag{40}$$

The eigenvalue spectrum is readily evaluated and one finds

$$\lambda(k) = -2\left[1 - \cos\left(\frac{\pi k}{N}\right)\right], \quad k = 1,2,\ldots,N. \tag{41}$$

Thus, the gap in the spectrum is given for $k=1$ with

$$\lambda(1) = -2\left[1 - \cos\left(\frac{\pi}{N}\right)\right]. \tag{42}$$

The other branches can be evaluated similarly, but lead to larger negative eigenvalues.

VIII. THE RUBINSTEIN-DUKE MODEL

The Rubinstein-Duke (RD) model is in a way complementary to the necklace model [see Fig. 1(b)]. In the RD model, the dynamical elements are reptons, blobs of monomers of the order of persistence length. A polymer chain is a string of reptons on a lattice with the constraint that reptons are either in the same cell or in neighboring cells. There are two types of links between reptons: the *slack* links connect two successive reptons in the same cell, and the *taut* links those in neighboring cells. A repton can hop to a neigboring cell if it does not leave an empty cell behind. Thus, the basic mechanism of motion is the interchange of taut and slack links. The elasticity of the chain is due to the slack links or the storage of length through occupation of cells with more than one repton. The integrity of the chain is guaranteed by requiring that a connected tube of cells is traced out by the chain. The natural embedding is in a lattice of dimension $d > 1$, but Duke [2] showed that it suffices to study the projection on the driving field direction. Thus, the dimension d becomes a parameter and the links are characterized by a three-valued variable: $y_i=0$ for a slack link, $y_i=1$ for a taut link in the direction of the field, and $y_i=-1$ for a taut link opposite to the field. The parameter d influences the ratio of slack to taut links, but not the universal properties of the chain, e.g., the exponents by which the renewal time and the drift velocity depend on the chain length. Thus, the model is often studied for $d=1$ although the one-dimensional version is rather artificial [8].

Sartoni and van Leeuwen [9] noted that the one-dimensional version of the RD model can be related to a model with two types of particles. Consider the superposition of two noninteracting one-dimensional systems of particles. The particles are called "+" viz. "−" and they hop according to the same rules as the vacancies in the necklace model. Thus, they are not allowed to occupy the same site, but since they are noninteracting, a plus particle may occupy the same site as a "−" particle. Therefore, each site can be in four states: empty, occupied by a "+" or a "−" or double occupied, which we indicate by "±." In total there are 4^N configurations. As the two systems do not interact, the probabilities of the combined system are the product of the probabilities of the two systems. Each of them can be treated with the matrix product expansion and yields identical expressions for, e.g., the drift velocity.

Having the freedom to choose the transition rates of the two systems independently, we drive the "+" particles in one direction and the "−" particles in the opposite direction with equal strength. Then, the "+" particles are identified with the links $y_i=1$ of the RD model and the "−" particles with the links $y_i=-1$. The 0 and ± state are both mapped on a slack link $y_i=0$. This gives a contraction of the 4^N configurations of the superposition to the 3^N configurations of the RD model. Knowing the probability distribution of the combined system, one can construct the transition rates in the master equation for the contracted system.

It will, in general, be quite an involved calculation. We illustrate it with a simplifying assumption, which is motivated by the following consideration. By reversing the driving field, the role of particles and vacancies are interchanged. The system is invariant under a transformation that maps the particles of one component onto the vacancies of the other component. In this map, an empty site, which is a vacancy state for both components, maps onto a doubly occupied site. Thus, as we will see, it is reasonable to equate the probabilities for an empty and a doubly occupied site.

Before we list the result for the transition rates of this constructed RD model, we note that the underlying combined system has a feature not present in the usual RD models. If we have a succession of two taut links, it can be a combination of the 0 and ± state. In that case, one of the particles of the "±" state can move to the empty site, thereby creating a pair of opposite taut links. This is called the creation of a hernia. It is not incorporated in the usual RD model, since it is a modification of the tube (consisting of the set of taut links). Similarly, the opposite process: a "+" and "−" particle meeting each other on the same site is the annihilation of a hernia. With this in mind we come to the following transitions: (i) Transitions of a slack-taut combination—The slack link corresponds to the two states of the "+" and "−" particle system. Each of them can interchange the taut and slack link, so the transition rate is the same. (ii) Transitions of a slack-slack combination—This corresponds to four states of the particles. Two of them cannot move: the 0, 0 and the ±, ± combination. The two others can create a hernia. Thus, the hernia creation rate is 1/2. (iii) Hernia annihilation—A hernia corresponds to a unique particle state and may develop into a slack-slack pair in two ways: the "+" particle may move or the "−" particle may. Thus, the hernia annihilation rate is 2. (iv) A slack end link—It has two particle configurations, 0 and "±," and both may move to the taut position, either by creating a particle in the state 0 or by annihilation of a particle in the state "±." Thus, the transition rate is 1. (v) A taut end link—It is a unique particle state that can transform itself in two ways in a slack state: by annihilation of the particle or by creation a particle of the other kind. Thus, it has a transition rate 2.

Therefore, the particle system maps onto a chain with asymmetric transition rates for the end reptons and for the creation and annihilation of hernias. The other moves are the same as in the RD model. The model with the above-listed transition rules will demonstrate the same behavior as the superposition of the "+" and "−" particles system with opposite driving fields. Although the map is based on a global and not a local symmetry, it is correct in the weak driving field limit. We have calculated, independently, the diffusion coefficient and the gap of this model by means of the DMRG method [10]. Figure 2(a) compares the drift velocity, in linear approximation, to the expression (29), divided by ϵ and refined with finite size effects (as can be deduced from [9])

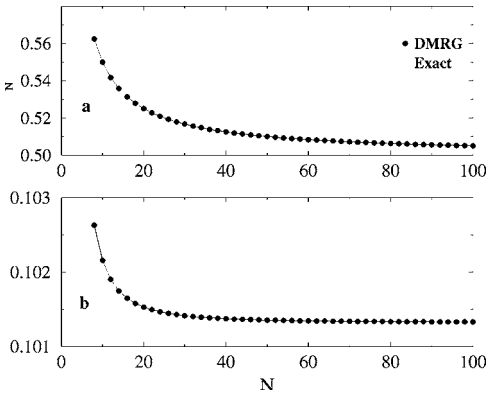

FIG. 2. The drift velocity for the RD model with hernias as a function of the length of the chain, together with the expression (43) for the necklace model (a). The drift velocity is divided by the trivial factor $B-B^{-1}$. (b) gives the renewal time of the RD chain with the expression (42) deduced from the necklace model.

$$v_N = \epsilon \frac{N+1}{2N}. \tag{43}$$

Note that the drift velocity of the RD model is twice the value of the necklace model, since there are two systems of particles moving independently.

Figure 2(b) does the same for the renewal time and compares it to the expression (42). As one observes, the agreement is perfect. Looking into the numbers, one has to conclude that the differences can be made arbitrarily small by making the DMRG calculation more accurate. It also proves that the DMRG method is amazingly accurate in calculating reptating polymer chains.

IX. DISCUSSION

We have given an exact solution for the stationary state of the necklace model introduced by Guidoni *et al.* [3]. Also an exact expression is given for the low-lying excitations (gap). This model mimics reptation of a chain in a one-dimensional system. It is not difficult to formulate the model in higher dimensions, but a chain in higher dimensions cannot be reconstructed from the position of the vacancies. Moreover, if such a system is driven by a field, the transition rates not only depend on the vacancy distribution, but also on the direction of the connecting links. Thus, one arrives at a model of the same complexity as the RD model. The model shows in $d=1$ not the characteristics of the slow reptation behavior. The drift velocity for long chains approaches a constant rather than decaying with the inverse powers of the length. The reason is that the model has no obstacles, which slow down the drift and the diffusion.

The solution can also be used for a special RD model in $d=1$, with the possibility of creation and annihilation of her-

nias. This shows the importance of the hernias in $d=1$. Although it is believed that hernia creation and annihilation is of minor importance in higher dimension, it plays a decisive role in $d=1$. Without the hernias as a move, the "+" taut links and the "−" taut links block each other and are driven toward each other. A hernia annihilation followed by a creation of a pair of taut links in the opposite order allows them to pass each other. Thus, the hernias, which are very abundant in $d=1$, effectively remove the obstacles, which are characteristic for the RD model.

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[1] M. Rubinstein, Phys. Rev. Lett. **59**, 1946 (1987).
[2] T. A. J. Duke, Phys. Rev. Lett. **62**, 2877 (1989).
[3] S. E. Guidoni, H. O. Mártin, and C. M. Aldao, Eur. Phys. J. E **7**, 291 (2002); Phys. Rev. E **67**, 031804 (2003).
[4] B. Derrida, M. Evans, V. Hakim, and V. Paquier, J. Phys. A **26**, 1493 (1993).
[5] G. Schütz, *Phase Transitions and Critical Phenomena*, Vol. 19, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 2001); Z. Torockzai and R. K. P. Zia, J. Stat. Phys. **87**, 545 (1997).
[6] T. Sasamoto, J. Phys. A **32**, 7109 (1999).
[7] R. A. Blythe, M. R. Evans, F. Colaiori, and F. H. L. Essler, J. Phys. A **33**, 2313 (2000).
[8] B. Widom, J.-L. Viovy, and A. D. Defontaines, J. Phys. I **1**, 1759 (1991).
[9] G. Sartoni and J. M. J. van Leeuwen, Phys. Rev. E **57**, 3088 (1998).
[10] S. R. White, Phys. Rev. Lett. **69**, 2863 (1992); *Density-Matrix Renormalization. A New Numerical Method in Physics*, edited by I. Peschel, X. Wang, M. Kaulke, and K. Hallberg, Lectures Notes in Physics Vol. 528 (Springer, New York, 1999); U. Schollwoeck, Rev. Mod. Phys. **77**, 259 (2005); A. Drzewiński, E. Carlon, and J. M. J. van Leeuwen, Phys. Rev. E **68**, 061801 (2003).